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Recent progress on the notion of global hyperbolicity

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Abstract. Global hyperbolicity is a central concept in Mathematical Relativity. Here, we review the different approaches to this concept explaining both, classical approaches and recent results. The former includes Cauchy hypersurfaces, naked singularities, and the space of the causal curves connecting two events. The latter includes structural results on globally hyperbolic spacetimes, their embeddability in Lorentz-Minkowski, and the recently revised notions of both causal and conformal boundaries. Moreover, two criteria for checking global hyperbolicity are reviewed. The first one applies to general splitting spacetimes. The second one characterizes accurately global hyperbolicity and spacelike Cauchy hypersurfaces for standard stationary spacetimes, in terms of a naturally associated Finsler metric.

1. Introduction and Notation

Global hyperbolicity is a central concept in Mathematical Relativity, which is involved in almost all global questions in this area, since the initial value problem to cosmic censorship (see for example [**42**]). The notion was introduced by Leray [**32**] in 1953, and developed in the *Golden Age of General Relativity* by Avez, Carter, Choquet-Bruhat, Clarke, Hawking, Geroch, Penrose, Seifert and others. However, some questions which affected basic approaches to this concept, remained unsolved in this epoch.

Concretely, the so-called *folk problems of smoothability* [**41**], affected the differentiable and metric structure of any globally hyperbolic spacetime M. Their ramifications include the possible embeddability of M in some Lorentz-Minkowski space $\mathbb{L}^N$ (in the spirit of Nash theorem), and other issues in the consistency of the *causal ladder* of spacetimes. Moreover, the (GKP) *causal boundary* [**24**] introduced a new ingredient for the causal structure of spacetimes, as well as a new viewpoint for global hyperbolicity. However, the lack of a full consistency for this boundary (especially in relation with the *conformal boundary*), remained as an open issue since this epoch.

Recently, these old issues have been revisited, and a full solution seems available now. Our purpose in this note is to give a brief account of both,

the old issues and the recent progress, focused on the notion of global hyperbolicity.

More precisely, in Section 2, the alternative definitions of global hyperbolicity in terms of topological elements of the spacetime (Cauchy hypersurfaces, absence of naked singularities) are explained. The equivalences are based in the central article by Geroch [**23**], and include a recent conceptual simplification in [**8**] (see Theorem 2.1). Section 3 is devoted to the implications of the folk problems of smoothability on the differentiable and metric structure of globally hyperbolic spacetimes (Theorem 3.1). Such a structure yields also a characterization of these spacetimes in terms of their embeddability in Lorentz-Minkowski (Proposition 3.2). In Section 4 original Leray's definition of global hyperbolicity is considered. This is expressed in terms of the space of (continuous) causal curves connecting two events. Such a space present some subtleties which are pointed out. In Section 5, the problem and recent solution to the notion of causal boundary is briefly explained. Then, a new characterization of global hyperbolicity is stated (Theorem 5.1). Moreover, the conditions under which the conformal boundary characterizes global hyperbolicity are also enounced (Theorem 5.2). The section is ended with a scheme about the different equivalent approaches to global hyperbolicity.

A different question is to determine, for a concrete spacetime, its possible global hyperbolicity. In the last two sections this question is studied for two families of spacetimes. The first one (Section 6), is the class of spacetimes which admit a smooth splitting type $\mathbb{R} \times S$ and such that the natural coordinate $t \in \mathbb{R}$ is a temporal function, being its levels $\{t_0\} \times S$ the natural candidates for Cauchy hypersurfaces (Theorem 6.1). The second one (Section 7), is the subclass of previous spacetimes $M = \mathbb{R} \times S$ which contains all standard stationary spacetimes. In these spacetimes, one characterizes exactly both, when the spacetime is globally hyperbolic and, in this case, when the slices $\{t_0\} \times S$ are Cauchy hypersurfaces. Such a characterization is expressed in terms of an auxiliary Finsler metric on S.

Throughout this paper, we will use standard notation in Causality, as in the books and reviews [**4, 22, 29, 34, 37, 40, 48**]. In particular, a spacetime will be a connected time-oriented n-manifold ($n \geq 2$), its chronological and causal relations are denoted $\ll, \leq$, resp., and these relations determine the chronological and causal futures and pasts $I^{\pm}(A), J^{\pm}(A)$ of any point or subset A of M. We also put:

$$\begin{gathered} I(p,q) := I^+(p) \cap I^-(q),, \quad J(p,q) := J^+(p) \cap J^-(q), \\ \uparrow P := I^+(\{x \in M : p \ll x, \ \forall p \in P\}) \\ \downarrow F := I^-(\{x \in M : x \ll q, \ \forall q \in F\}) \end{gathered}$$

for any $p, q \in M$ and $P, F \subset M$. Timelike and causal curves are regarded as smooth (C^1 is enough) curves with timelike or causal derivative, except in Section 4, where *continuous causal* curves are explicitly considered.

2. Topological equivalences on the manifold

The simplest way to understand global hyperbolicity relies on the interplay between the causality of the spacetime and some topological elements of the manifold. Concretely, recall the following ones:

- *Cauchy hypersurface:* subset $S \subset M$ which is crossed exactly once by any inextensible timelike curve.

  Then, S must be an embedded topological hypersurface and must be also crossed by any inextensible causal curve γ [**21, 37**]. However, such a γ may intersect S not only in a point but also along a compact interval of its domain. If causal curves cannot intersect S in more than one point, then S is an *acausal Cauchy hypersurface*.

  Easily, the existence of a Cauchy hypersurface S implies that M is homeomorphic to $\mathbb{R} \times S$, and all Cauchy hypersurfaces are homeomorphic.
- *Time function:* continuous function $t : M \to \mathbb{R}$ which increases strictly on any future-directed causal curve.

  If, moreover, the levels t =constant are Cauchy hypersurfaces, then t is a *Cauchy time function*. By convenience, all Cauchy functions will be assumed onto (this is not restrictive because, otherwise, a re-scaling of t will fulfill this property).
- *Absence of naked singularities*: this means that $J(p, q)$ is compact for all p, q in M.

  In fact, if $J(p, q)$ is non-compact for some p, q, then one can check the existence of a future-directed causal curve ρ starting at p and contained in $J(p, q)$, with no endpoint in the closure $\overline{J(p, q)}$. Moreover, this curve is always visible from q, i.e., any point $\rho(s)$ can be joined with q by means of a future-directed causal curve. Summing up, in this case one can say that a (physical) *naked singularity* appears between p and q (see also [**15**]).

The connections among these elements are summarized in the following result.

Theorem 2.1. *For a spacetime M, the following items are equivalent:*

1. *M is causal and does not have naked singularities.*
2. *M is strongly causal and does not have naked singularities.*
3. *M admits a Cauchy time function t.*
4. *M admits a Cauchy hypersurface S.*

M is called globally hyperbolic when such equivalent items holds.

The following comments on Theorem 2.1 are in order. Item (2) is a typical definition of global hyperbolicity, which is used in standard books such as [**4, 29, 37, 40, 48**]. Item (1) is an even simpler definition in [**8**]. In fact, (1) $\Rightarrow$ (2) because the assumptions (1) imply that M is *causally simple* (as $J^{\pm}(p)$ is closed for all p) and, therefore, strongly causal, as required. It

is worth pointing out that, under strong causality, the compactness of the closures $\overline{J(p,q)}$ is enough to ensure global hyperbolicity [**4**, Lemma 4.29], but under causality such a property is not enough (Carter's example [**29**, p. 195], see [**20**]).

The implication (2) $\Rightarrow$ (3) relies on the following celebrated idea by Geroch [**23**] (this author use the definition of global hyperbolicity explained in Section 4, so, see also [**44**]). First, one takes any finite measure m on M associated to some Riemannian metric –or, with more generality, one can take also any *admissible measure* in the sense of Dieckmann [**16**]). Then, define the past and future volume functions $t^{\pm}: M \to \mathbb{R}$, as $t^-(p) := m(I^-(p)), t^+(p) := -m(I^+(p))$. These functions are time functions for *causally continuous* spacetimes (a class of spacetimes more general than the causally simple ones and, so, which includes all the globally hyperbolic spacetimes). Moreover, if $\gamma : (a,b) \to M$ is an inextensible future-directed causal curve, one has:

$$\lim_{s\to b} t^+(\gamma(s)) = 0 = \lim_{s\to a} t^-(\gamma(s)) \quad \lim_{s\to a}(-t^+(\gamma(s))), \lim_{s\to b} t^-(\gamma(s)) > 0$$

so that $t(z) = \log\left(-t^-(z)/t^+(z)\right)$ satisfies:

$$\left.\begin{array}{l} \lim_{s\to b} t(\gamma(s)) = \infty \\ \lim_{s\to a} t(\gamma(s)) = -\infty \end{array}\right\} \Longrightarrow \text{levels } t = \text{const. are Cauchy}$$

i.e., t is the required Cauchy time function.

As the implication (3)$\Rightarrow$ (4) is trivial, all the equivalences hold if one proves (4)$\Rightarrow$ (1). This follows by using nowadays standard arguments on limit curves [**4**] or quasilimits [**37**]. In fact, the absence of naked singularities follows from the compactness of the sets type $J^-(q) \cap J^+(S), q \in M$ (and to prove causality is now an exercise simpler than strong causality), see [**4, 21, 37**].

3. Smoothability and structural results

In previous section, the involved elements were defined at a topological level, and we was not worried about its differentiability. But, of course, this property will turn out essential for applications. So, we will say that a Cauchy hypersurface or time function is *smooth* if it is as differentiable as allowed by the order of differentiability of the spacetime. However, smoothness will not be enough for relevant applications and, so, we state the following notions:

- *Spacelike Cauchy hypersurface:* a smooth Cauchy hypersurface S such that all the tangent hyperplanes T_pS, $p \in S$, are spacelike.

  Necessarily, S is then acausal, but notice that a smooth acausal Cauchy hypersurface may be non-spacelike, as the tangent hyperplanes may be degenerate. From a technical viewpoint, the initial value problem starts typically with a smooth Riemannian manifold

which will be a posteriori a spacelike Cauchy hypersurface of the evolved spacetime.

Easily, the existence of a smooth Cauchy hypersurface S implies that M is diffeomorphic to $\mathbb{R} \times S$, and all smooth Cauchy hypersurfaces are diffeomorphic.

- *Temporal function:* smooth time function $t : M \to \mathbb{R}$ such that its gradient ∇t is timelike –or, equivalently, a smooth function t with past-directed timelike gradient. If, additionally, the levels t =constant are Cauchy hypersurfaces (necessarily spacelike ones), then t is a *Cauchy temporal function*, and, again, t will be assumed onto with no loss of generality.

  The existence of a Cauchy temporal function t is especially interesting, because it is equivalent to the existence of a global orthogonal splitting $M \equiv (\mathbb{R} \times S, g)$, where g can be written as

$$g = -\beta dt^2 + g_t, \tag{3.1}$$

  being β (the *lapse*) a function on $\mathbb{R} \times S$, and (under a natural identification) g_t a Riemannian metric on each slice $\{t\} \times S$ varying smoothly with $t \in \mathbb{R}$.
- *Steep temporal function*: a temporal function t whose gradient satisfies $|\nabla\tau|^2 (:= -g(\nabla t, \nabla t)) \geq 1$.

  The existence of a steep temporal function is interesting for a spacetime M, as it solves the problem of its isometric embeddability in some $\mathbb{L}^N$ (in the spirit of Nash' theorem [**36**]), see [**35**]. In fact, it was noticed by Greene [**25**] and Clarke [**14**] that any semi-Riemannian (or even degenerate) manifold can be isometrically immersed in some semi-Euclidean space $\mathbb{R}^N_s$ of sufficiently big dimension N and index s. The problem is a bit subtler for $s = 1$, but a simple argument in [**35**] shows that a spacetime can be isometrically embedded in $\mathbb{L}^N$ if and only if it admits a steep temporal function.

  Moreover, a spacetime which admits a *steep Cauchy temporal function* can be split as in (3.1) with a bounded lapse function, as $\beta = |\nabla\tau|^{-2}$.

The equivalence of previous elements with global hyperbolicity can be summarized as follows.

Theorem 3.1. *For a spacetime M, any of the following items is equivalent to global hyperbolicity:*

1. *M admits a spacelike Cauchy hypersurface.*
2. *M admits a Cauchy temporal function t or, equivalently, M admits a splitting $M \equiv (\mathbb{R} \times S, g)$ with g as in (3.1).*
3. *M admits a steep Cauchy temporal function or, equivalently, a global splitting $M \equiv (\mathbb{R} \times S, g)$ where g adopts the form (3.1) with $\beta < 1$.*

The following comments on this theorem are in order. The so-called *folk problems of smoothability* consist in the question whether the alternative definitions of global hyperbolicity in Theorem 2.1 imply the items (1) or (2) in Theorem 3.1 (or, with more generality, if certain type of causally constructed continuous elements can be obtained also in a smooth way, with an appropriate causal character). The first problem (glob. hyp. $\Rightarrow$ Theorem 2.1(1)) was posed explicitly in [**41**, p. 1155] and solved in [**5**]. The second one (glob. hyp. $\Rightarrow$ Theorem 2.1(2)) was solved in [**6**]. Moreover, here the question whether any spacetime which admits a time function must admit a temporal function too, is answered affirmatively. This question affected the consistency of the two classical definitions of *stable causality* and, thus, the structure of the so-called *causal hierarchy of spacetimes* (see the review [**34**] for full details).

The difficulty in the solution of these problems relied in two facts:

(a) Notice that the volume functions $t^{\pm}$ in the proof of Geroch's theorem are continuous in the globally hyperbolic case, and there exists a big freedom of admissible measures in order to construct them. Nevertheless, in general one cannot expect that, for example, if $t^{\pm}$ is only a continuous time function constructed from some admissible measure then, by changing this measure, the new functions $t^{\pm}$ will be smooth. In fact, $t^{\pm}$ are time functions if and only if the spacetime is causally continuous [**35**], that is, $t^{\pm}$ are not continuous if the spacetime is only stably causal. Nevertheless, in such a spacetime, the existence of a time function is ensured thanks to an original argument by Hawking [**28**]. Of course, to change the admissible measure is useless for this argument, as $t^{\pm}$ are always non-continuous.

(b) The smoothability problems not only affect smoothability, but also the causal character of the involved elements. If, for example, some sort of smoothing procedure (say, some type of convolution) would yield a smooth Cauchy hypersurface S, one would have to study still the case when S is degenerate. And this would be delicate, as a small perturbation of S (in order to get a spacelike hypersurface) might spoil the Cauchy character.

Due to these two difficulties, the proofs in [**5, 6**] are based in the construction of some sort of semi-local temporal functions and a systematic process of sum, very different to the arguments in the proof of Theorem 2.1. However, the Cauchy temporal function constructed by Geroch [**23**], is required in order to start the process.

It is also worth pointing out that the Cauchy temporal function t can be chosen such that any prescribed spacelike hypersurface S is one of its levels (and S can be chosen such that any prescribed acausal compact spacelike

submanifold with boundary is included in S), [**7**]. This contributes to the consistency of the usual procedures in Mathematical Relativity[1].

The item (3) becomes relevant in both ways: it is a refinement of the structural decomposition (3.1) for any globally hyperbolic spacetime, and it implies the isometric embeddability of all globally hyperbolic spacetimes in Lorentz-Minkowski. This embeddability had been already claimed by Clarke [**14**]; however, his proof was affected by the folk problems of smoothability. The proof in [**35**] is based in a constructive procedure of a steep Cauchy temporal function, which also starts at Geroch's Cauchy time function. This construction is independent and easier than the one in [**6**]. However, it is carried out specifically for globally hyperbolic spacetimes. Thus, in principle, it cannot be used to construct a temporal function in any stably causal spacetime.

Finally, it is worth pointing out that, for a spacetime which admits a temporal function but it is not globally hyperbolic, the existence of a steep temporal function can be lost or gained by changing conformally the metric [**35**]. So, one has:

Proposition 3.2. *Let M be a spacetime. Then:*

(A) M is globally hyperbolic if and only if all the spacetimes in its conformal class are isometrically embeddible in some Lorentz-Minkowski space $\mathbb{L}^N$ of sufficiently high dimension N.

(B) M is stably causal (i.e., M admits a temporal function) if and only if some representative of its conformal class can be isometrically embedded in some Lorentz-Minkowski space $\mathbb{L}^N$.

4. The space of continuous causal curves

Historically, the notion of global hyperbolicity was introduced by Leray [**32**] starting at the the space of causal curves $C_{p,q}$ connecting two points, in a wider frame for hyperbolic equations. This first notion was developed fast (for example, see [**2, 13, 27, 38, 46**]). It is worth pointing out some technicalities on this space.

First, all these curves will be taken reparameterized in the same interval, namely $I = [0, 1]$, so that the compact-open topology will be assumed in $C_{p,q}$. However, sequences of (smooth) causal curves will have non–smooth limits in a natural way, and these limits must be regarded as causal too. So, a *(future-directed) continuous causal curve* $\gamma : I \to M$ is defined as a (continuous) curve which, for each convex neighbourhood[2] $U \subset M$, satisfies: if $t, t' \in I, t \leq t'$ with $\gamma([t, t']) \subset U$, then $\gamma(t) \leq_U \gamma(t')$ (where $\leq_U$ denotes the causal relation in U, regarded as a spacetime). One can check that, for a

[1]For example, in the initial value problem, one could conceive the following situation: the initial hypersurface is a Cauchy hypersurface of the evolved spacetime, but it is not a slice of any Cauchy temporal function. In this case, well-posed initial data on S would imply structural restrictions in the evolved spacetime.

[2]i.e., U is a (starshaped) normal neighbourhood of all its points (see [**37**, p. 129]).

(continuous) curve γ defined on I and non locally constant around any point $t_0 \in I$, the following equivalence holds [**11**, Appendix B]: γ is continuous causal iff γ is H^1 and $\dot\gamma(s)$ is a future–directed causal vector for almost all $s \in I$ –in particular, γ is Lipschitzian[3]. The space $C_{p,q}$ is then the set of all these continuous causal curves which connect p with q, endowed with the compact-open topology (or equivalently, with the topology of uniform convergence).

However, one has still the problem that all the reparameterizations of a single curve $\gamma \in C_{p,q}$ yields a non-compact subset. Following Choquet-Bruhat [**13**], one can fix some auxiliary Riemannian metric h and consider just the subset:

$$C^h_{p,q} = \{\gamma \in C_{p,q} : h(\dot\gamma, \dot\gamma) = c \text{ (constant) a.e.}\}.$$

We emphasize that, even in $\mathbb{L}^N$, the space $C^h_{p,q}$ is not compact whenever $p \ll q$ (this, as well as other subtleties along this section, have been studied in [**9**]). However, the closure of $C^h_{p,q}$ in $C_{p,q}$ will be compact, which will be enough for our purposes.

Theorem 4.1. *For a spacetime M, the following items are equivalent:*

(1) *M is globally hyperbolic.*
(2) *For each $p, q \in M$, the set of all the continuous causal curves $C^h_{p,q}$ (parametrized on $[0, 1]$ at constant speed for an auxiliary Riemannian metric h) has a compact closure in the space of all the continuous causal curves $C_{p,q}$ endowed with the compact-open topology.*

In fact, (2) $\Rightarrow$ (1) is now straightforward, as one can check easily that (2) implies both the absence of naked singularities and causality (remarkably, the reasoning by Choquet-Bruhat in [**13**] proved directly strong causality). The converse can be proved by using known tools of limit curves.

Many of the bothering subtleties for $C_{p,q}$ come from the reparameterizations of the curves. In order to avoid this, we will call the image of a continuous causal curve a (causal) *path*. Notice that, if a causal curve α is closed, then going two rounds along it we obtain a new curve α^2. Obviously, α and α^2 yield the same path, but none of them is a reparameterization of the other one. Such paths will be excluded by considering causal spacetimes. In these spacetimes, a path can be regarded also as the class of a continuous causal curve in $C^h_{p,q}$ up to a reparameterization. A natural topology for paths is the C^0 one –namely, a path ρ is the limit of a sequence $\{\rho_n\}_n$ if any open set $U \subset M$ which contains ρ contains also all but a finite number of ρ_n, see [**4**]. Good properties on convergence for the C^0 topology appear in strongly causal spacetimes. However, as the endpoints of the paths in $C_{p,q}$ are fixed, causality will be enough for the following characterization.

Theorem 4.2. *For a causal spacetime M, the following are equivalent:*

[3]In this Lorentzian setting, concepts such as H^1 or Lipschitzian (or uniform convergence below) can be regarded as those for any auxiliary Riemannian metric h.

(1) *M is globally hyperbolic.*
(2) *For each $p, q \in M$, the space of continuous causal paths $C_{p,q}^{path}$ (i.e., the set of the images of continuous causal curves in $C_{p,q}$) endowed with the C^0 topology, is compact.*

In fact, (2) $\Rightarrow$ (1) can be found in Geroch's [**23**], and the converse follows from known properties of the C^0 convergence which goes back to [**46**].

Finally, it is worth pointing out that, because of the compactness properties of $C_{p,q}$, and the lower continuity of the energy functional on $C_{p,q}$, one can prove easily the following Avez-Seifert property [**2, 46**]: *if two points $p \neq q$ of a globally hyperbolic spacetime M are causally related ($p < q$), then they can be connected by means of a causal geodesic with length equal to the time-separation (Lorentzian distance) $d(p,q)$.* In particular, $d(p,q)$ is always finite in globally hyperbolic spacetimes. However, for non-globally hyperbolic spacetimes, the finiteness of $d(p,q)$ at some p, q, will be lost for some representatives of the conformal class [**4**, Th. 4.30]. Summing up:

Proposition 4.3. *For a strongly causal spacetime M, the following properties are equivalent:*

(1) M is globally hyperbolic.

(2) The Lorentzian distance d is finite valued for all the spacetimes in the conformal class of M.

5. Causal and conformal boundaries

Among the notions of boundary for a spacetime, the conformal and causal ones are the most useful and promising in Mathematical Relativity. Global hyperbolicity is closely related to the properties of these boundaries; in fact, it is commonly claimed that a spacetime is globally hyperbolic when the causal or conformal boundary does not contain a *timelike point*. Nevertheless, both boundaries have presented problems of consistency, which only recently have been solved. So, we summarize very briefly these problems and how global hyperbolicity can be characterized in terms of these boundaries. We refer to [**18**] for exhaustive discussions and references.

The notion of causal boundary $\partial_c M$ for a spacetime M was introduced by Geroch, Kronheimer and Penrose [**24**] (*GKP boundary*). The idea was to attach a boundary $\partial_c M$ to to any strongly causal spacetime so that each inextensible future-directed (resp. past-directed) timelike curve will have an endpoint in $\partial_c M$. The initial idea was that two such future-directed (resp. past-directed) timelike curves $\gamma, \tilde{\gamma}$ must reach the same point when their chronological pasts (resp. future) coincide, i.e., $I^-(\gamma) = I^-(\tilde{\gamma})$ (resp. $I^+(\gamma) = I^+(\tilde{\gamma})$). So all these past (resp. future) sets or *TIPs* (resp. *TIFs*) would be regarded as boundary points. However, two problems appear. The first one is that, sometimes, it is natural to expect that both, an inextensible future-direct timelike curve and a past-directed one, will have the same endpoint, i.e., a TIP and a TIF would be identified as the same boundary

point[4]. The second one is to topologize the completion –so that one can check precisely when a sequence or curve in M converges to a point in $\partial_c M$. These two problems are closely related, and yielded a hard *identification problem*, studied by many authors shortly after the seminal GKP paper (see, for example, [**10, 31, 47**]). The main difficulty for this problem was that, apparently, there were many possible choices of both, identifications and topologies. Nevertheless, no one of them seemed to be naturally consistent in the following sense: if M is a simple open subset of Lorentz-Minkowski $\mathbb{L}^N$, $\partial_c M$ must agree with the topological boundary of M in $\mathbb{L}^N$ –or, at least, a satisfactory reason which justifies the discrepancy must be provided.

A critical review on the different attempts to solve this problem can be found in [**45**]. Here, we point out just the following elements of the solution provided in [**18**], which takes into account the recent progress in [**17, 26, 33**]. The causal boundary $\partial_c M$ is composed by *timelike* points and *non-timelike* points. The former are the pairs type (P, F) where P is a TIP, F is a TIF and they are S-related, i.e., P is included and is maximal[5] in the common past $\downarrow F$ of F and, viceversa, F is included and is maximal in the common future $\uparrow P$. The non-timelike points are pairs type $(P, \emptyset)$ or $(\emptyset, F)$, where P (resp. F) is a TIP (resp. TIF) which is not S-related with any TIF (resp, TIP). The spacetime itself is also regarded as the set of all the pairs $M \equiv \{(I^+(p), I^-(p)) : p \in M\}$ so that one has naturally the causal completion $\overline{M} = M \cup \partial_c M$, composed by pairs of subsets of M. The chronological relation $\ll$ is extended to a natural chronology $\overline{\ll}$ in $\overline{M}$, namely: $(P, F)\overline{\ll}(P', F')$ if $F \cap P' \neq \emptyset$. Moreover, there exists a natural (but subtle) way to topologize $\overline{M}$.

So, if $(P, F) \in \partial_c M$ is a timelike point, one has $p\overline{\ll}(P, F)\overline{\ll}q$ for any $p \in P$, $q \in F$. Nevertheless, for a non-timelike point, say, $(P, \emptyset)$, there is no other $(P', F') \in \overline{M}$ which lies in its $\overline{\ll}$–chronological future. It is not difficult to check that the existence of a timelike point is equivalent to the existence of a naked singularity (see [**18**] for exhaustive details). So, one has:

[4]Think, for example, in $M = \{(x, t) : x > 0\} \subset \mathbb{L}^2$. Each $(0, t) \in \mathbb{L}^2$ yields naturally a TIP, $P = I^-((0, t)) \cap M$, and a TIF, $F = I^+((0, t)) \cap M$, which should be regarded as a unique boundary point.

[5]Maximal in the sense that no other TIP P' satisfies $P \subsetneq P' \subset\downarrow F$.

Theorem 5.1. *For a strongly causal spacetime*[6] *M, the following properties are equivalent:*

(1) *M is globally hyperbolic.*
(2) *The causal boundary $\partial_c M$ of M does not admit a timelike point, i.e., any pair $(P, F) \in \partial_c M$ satisfies either $P = \emptyset$ or $F = \emptyset$.*

About the conformal boundary, the following remarks are in order. First, it is the most commonly used boundary in Mathematical Relativity: the Penrose-Carter diagrams (or typical concepts such as asymptotic flatness [**1, 48**]) are stated in terms of a conformal boundary, see [**29**]. However, this boundary is just an *ad hoc* construction for some classes of spacetimes. In fact, it is defined by using an open conformal embedding $i : M \hookrightarrow M_0$ and taking the topological boundary of the image $\partial_i M := \partial(i(M))$. However, there is no a general recipe which says when two such embeddings i, j will yield conformal boundaries $\partial_i M, \partial_j M$ which are isomorphic in some natural sense –or when a given spacetime will admit some useful open conformal embedding.

As emphasized in [**18**], the best we can say is that, under some hypotheses, the conformal boundary $\partial_i M$ (or some part of it) will agree with the causal one $\partial_c M$. Then, $\partial_i M$ will reflect intrinsic properties of M. In this case, $\partial_i M$ may be very useful from a practical viewpoint, as it may be much easier to compute than $\partial_c M$.

A first condition for the identification of both boundaries is the following one. The open conformal embedding $i : M \hookrightarrow M_0$ must be *chronologically complete*, i.e, if γ is a future-directed inextensible curve in M then the curve $i \circ \gamma$ (which is necessarily future-directed and timelike in M) must have a future endpoint in $\partial(i(M))$ –and analogously for past-directed curves. We will not go through more details on the conditions for the identifications of both boundaries, which are detailed in [**18**]. Simply, we recall the following characterization of global hyperbolicity, when one can say that the conformal boundary is C^1:

Theorem 5.2. *Let M be a spacetime which admits an open conformal embedding $i : M \hookrightarrow M_0$ in a strongly causal spacetime M_0 such that: (i) $i(M)$ is an open subset of M_0 with C^1 boundary, and (ii) i is chronologically complete. Then, the following properties are equivalent:*

(1) *M is globally hyperbolic.*

[6] An important reason to impose strong causality is to ensure that the completion $\overline{M}$ will have a natural satisfactory topology –namely, the topology on M will be the Alexandrov one, generated by the sets type $I(p, q)$, and this topology coincides with the manifold topology only in strongly causal spacetimes. In principle, the notion of causal boundary as a point set endowed with a extended chronological relation would make sense for causal spacetimes. Harris [**26**] considered even a more general notion of chronological set. However, some subtleties appear when one likes to ensure that, for any inextensible future-directed timelike curve γ, the set $I^-(\gamma)$ is truly a TIP, in the sense of *terminal indecomposable set* introduced in [**24**].

(2) *The conformal boundary* $\partial_i M := \partial(i(M))$ *of* M *does not admit a* timelike point*, i.e., no* $z \in \partial_i M (\subset T_z M_0)$ *admits as tangent space* $T_z(\partial_i M)$ *an hyperplane with Lorentzian signature.*

Moreover, in this case the causal boundary $\partial_c M$ *is naturally identified with the conformal one* $\partial_i M$*.*

We remark that, in this theorem, the abstract notion of timelike point for the causal boundary, is translated in the more palpable notion of a point in the conformal boundary with timelike tangent hyperplane. It is very easy to prove that (1) $\Rightarrow$ (2) (in fact, the negation of (2) at a point z, yields directly a naked singularity, and only the local properties around z are relevant for this). However, all the properties of the embedding must be carefully used in order to obtain the reversed implication, see [**18**].

The equivalences between all previous approaches to global hyperbolicity are summarized in the figure in the next page.

6. Checking global hyperbolicity in general splitting spacetimes

Taken into account the Cauchy splitting (3.1), we can wonder, conversely, when a spacetime splitted as in (3.1) admits as Cauchy hypersurfaces the levels $t \equiv$ constant. Even more, the applicability will be bigger if we admit also mixed terms between the parts in $\mathbb{R}$ and S. This was studied systematically in [**43**], and the results are summarized next.

Let M be a spacetime which splits smoothly as $M = \mathbb{R} \times S$, being its metric g at each $z = (t, x) \in M$:

$$g((\tau,\xi),(\tau,\xi)) = -\beta(z)\tau^2 + 2 < \delta(z), \xi > \tau + < \alpha_z(\xi), \xi >, \tag{6.1}$$

for all $(\tau, \xi) \in T_z M \equiv \mathbb{R} \times T_x S$. Here, β is a positive function on M, $< \cdot, \cdot >$ denotes a fixed auxiliary Riemannian metric on S (with associated norm $\| \cdot \|$ and distance $d(\cdot, \cdot)$), α_z is a symmetric positive operator on $T_x S$ and $\delta(z) \in T_x S$, all varying smoothly with z. The minimum eigenvalue of α_z will be denoted $\lambda_{min}(z)$; notice that the eigenvalues of α_z vary (a priori just) continuously with z.

No more generality would be obtained if we put $M = I \times S$ for some interval $I \subset \mathbb{R}$, as a re-scaling of the projection $t : M \to I$ would reduce this case to the former one. Analogously, we will assume that $< \cdot, \cdot >$ is *complete* with no loss of generality (we are free to choose any auxiliary Riemannian metric, and different choices would redefine δ and α). It is straightforward to check that, under our assumptions, g must be Lorentzian, and we assume that the future time-orientation is defined by ∂_t.

Now, notice that an inextensible future-directed causal curve $\gamma(s) = (t(s), x(s))$ can be reparameterized by the coordinate t, and will cross all the slices $S_t = \{t\} \times S_0$ if and only if the curve $t \to \bar{x}(t) = x(s(t))$ can be continuously extended to any finite value of t. Thus the slices S_t will be Cauchy if the curves type $\bar{x}(t)$ which come from a causal curve have finite $\langle \cdot, \cdot \rangle$-length for finite values of t. Then, a computation shows [**43**, Sect. 3]:

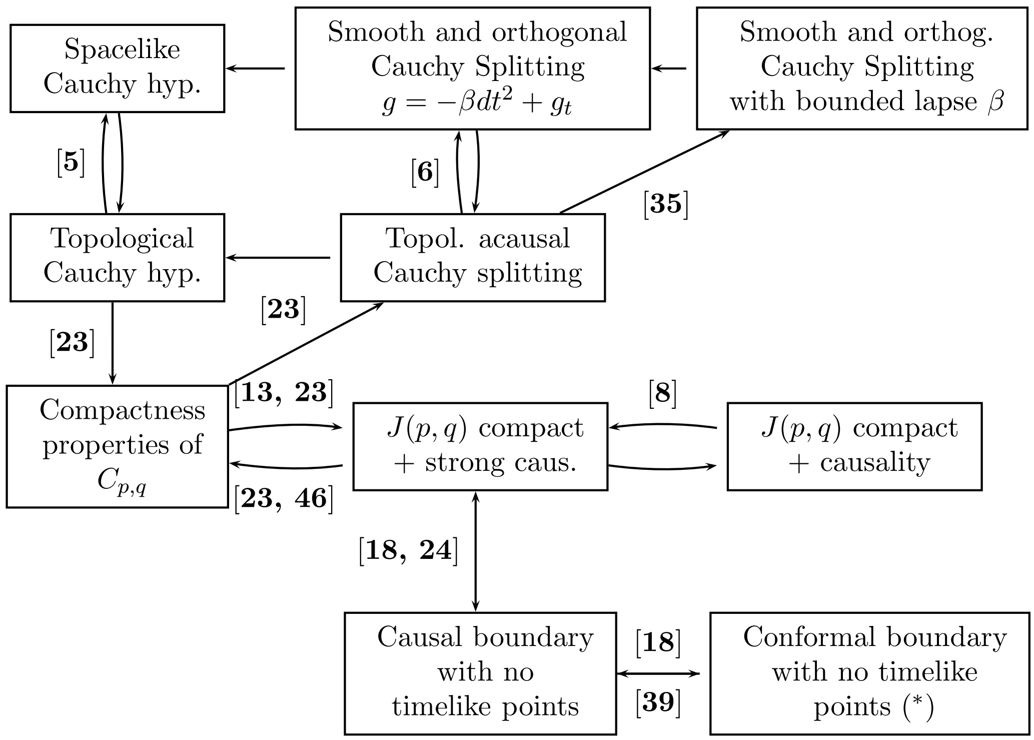

Figure. Summary of both, the classical and revisited notions of global hyperbolicity:

In the first row, the characterizations of global hyp. involve differentiability and metric properties. They imply trivially the second row. The converses solve the so-called "folk problems" of smoothability, first pointed out in [**41**].

The equivalence between the second and third rows relies on the fundamental theorem by Geroch [**23**]. He used a notion of global hyp. based on the space of curves $C_{p,q}$ introduced by Leray [**32**]. The simplification of this definition in terms of $J(p,q)$ in the third row, become widely convenient both, technically (see, in general, Hawking and Ellis [**29**]) and conceptually (the compactness of $J(p,q)$ can be understood as the absence of naked singularities).

The main problem to prove the equivalences between the third and the fourth rows, was to find consistent definitions for both, the causal [**24**] and the conformal [**39**] boundaries. The drawn equivalence with a property of the causal boundary (as defined in [**18**]), is completely general. Nevertheless, the conformal boundary can be defined only in some cases.

(*) The equivalence for the conformal boundary holds when a chronologically complete open conformal embedding with C^1 boundary in a strongly causal spacetime (according to the definitions in [**18**]) exists.

Theorem 6.1. *For each positive integer n, put $M[n] = [-n, n] \times S \subset M$, and assume that there exists a smooth function F_n on S for each n such that:*

(1) *the following inequality holds for all $(t, x) \in M[n]$:*

$$\frac{\| \delta \| + (\lambda_{min}\beta + \| \delta \|^2)^{1/2}}{\lambda_{min}}(t, x) \leq F_n(x),$$

(2) *the conformal metric $< \cdot, \cdot >_n = < \cdot, \cdot > /F_n^2$ on S is also complete.*

Then each slice $S_t = \{t\} \times S$ is a Cauchy surface.

In particular, this happens if the following constants M_n satisfy:

$$M_n = \mathrm{Sup}\{\frac{\| \delta(z) \|}{\lambda_{min}(z)\, d_0(x)}, \sqrt{\frac{\beta(z)}{\lambda_{min}(z)\, d_0^2(x)}} : z = (t, x) \in M[n], d_0(x) > 1\} < \infty$$

where $d_0 : S \to \mathbb{R}$ is the $< \cdot, \cdot >$–distance function to some (and then any) fixed point $x_0 \in S_0$.

Remark 6.2. (1) Of course, $\lim_n M_n = \infty$ is allowed.

(2) The hypothesis $d_0(x) > 1$ in the definition of M_n is imposed because the relevant behavior of the metric elements occurs at infinity. Equally, one can assume $d_0(x) > C$ for some convenient constant C. Notice that, at any case, the completeness of $\langle \cdot, \cdot \rangle$ implies the compactness of the closed balls.

(3) In particular, if S is compact then all the slices $\{t\} \times S$ are Cauchy with no further assumption.

It is not difficult to sharpen these estimates by using subtler bounds (say, bounding the elements of the splitting by suitable radial functions with finite integral, so that $\bar{x}(t)$ will not be divergent). We will not go into this general possibility, but we will see next that a clean characterization holds for stationary spacetimes.

7. Global hyperbolicity in standard stationary spacetimes

Among the spacetimes which admit an expression as in (6.1), appear the standard stationary ones. In fact, they are defined by such a splitting when all the elements β, δ, α are independent of the coordinate t, i.e., the timelike vector field ∂_t is Killing[7]. In this case, we can define a metric $g_0 = < \alpha(\cdot), \cdot >$ on S, which is independent of t and may be incomplete. That is, for a standard stationary spacetime the metric (6.1) turns out into:

$$g((\tau, \xi), (\tau, \xi)) = -\beta(x)\tau^2 + 2g_0(\delta(x), \xi)\tau + g_0(\xi, \xi), \tag{7.1}$$

for all $(\tau, \xi) \in T_zM \equiv \mathbb{R} \times T_xS$. If, moreover, $\delta \equiv 0$ then M is called standard static; that is, in this case $g = -\beta dt^2 + g_0$, with natural identifications.

Theorem 6.1 yields directly the following consequence.

[7]Recall that any distinguishing spacetime with a complete timelike Killing vector field K can be written as a standard stationary [**30**] (and, then, $K \equiv \partial_t$). The proof of this result is also a consequence of the solution of the folk problems of smoothability.

Corollary 7.1. *Assume that (M, g) is standard stationary, according to (7.1). If g_0 is complete and*

$$\text{Sup}\{\frac{\parallel \delta \parallel}{d_0}(x), \frac{\sqrt{\beta}}{d_0}(x) : x \in S, d_0(x) > 1\} < \infty, \tag{7.2}$$

then the spacetime is globally hyperbolic and all the slices S_t are Cauchy hypersurfaces.

In particular, (7.2) holds if there exist constants a, b, c, d such that, for all x outside some compact subset:

$$\parallel \delta(x) \parallel < ad_0(x) + b, \text{ and } \sqrt{\beta}(x) < cd_0(x) + d.$$

Notice that these results yield a rough estimate on the different elements in the standard splitting so that the slices S_t will be globally hyperbolic. However, it is clear that these estimates are only sufficient conditions. For example, in the static case $g = -\beta dt^2 + g_0$, Corollary 7.1 imposes that g_0 is complete and β grows at most quadratically with the distance at infinity. Notice that both conditions are sufficient to ensure that g_0/β is complete. However, as Causality is conformally invariant, one can study directly the global hyperbolicity of $g^* = -dt^2 + g_0/\beta$. Then, Corollary 7.1 applied to g^* shows that the completeness of g_0/β ensure that the slices $\{t\} \times S$ are Cauchy (for g^* and, thus, for g). Moreover, a simple computation [**4**, Th. 3.67] shows that the converse holds and, additionally, that g is globally hyperbolic only if the hypersurfaces S_t are Cauchy.

One can generalize this sharpened results to standard stationary spacetimes as follows. We will characterize both, when the slices S_t are Cauchy hypersurfaces and also when M is globally hyperbolic, even if the slices S_t are not Cauchy. This characterization is obtained in terms of the *Fermat metric* associated to the standard stationary splitting. This is a Finsler metric on S of Randers type. The characterization, as well as many consequences, is developed in detail in [**12**]. Here, we only describe it very briefly.

Given the standard stationary splitting (7.1) the associated Fermat metric on S is defined as:

$$F(v) = \frac{1}{\beta} g_0(v, \delta) + \sqrt{\frac{1}{\beta} g_0(v, v) + \frac{1}{\beta^2} g_0(v, \delta)^2}, \quad \forall v \in T_pS, p \in S. \tag{7.3}$$

This is a type of non-reversible Finsler metric[8], and it induces a (non-necessarily symmetric) distance d_F on S. Such a d_F also induces forward and backward closed balls, defined respectively as:

$$\bar{B}^+(x, r) = \{y \in M : d_F(x, y) \leq r\}, \quad \bar{B}^-(x, r) = \{y \in M : d_F(y, x) \leq r\}.$$

Analogously, one has forward and backward Cauchy sequences, and forward and backward completeness. We can also define the symmetrized distance

[8]Non-reversible means $F(v) \neq F(-v)$, in general. We remind that a Finsler metric yields a non-symmetric norm with strongly convex balls in each tangent space T_xS smoothly varying with $x \in S$, see for example [**3**].

$d_s(x,y) = (d_F(x,y)+d_F(y,x))/2$ and the corresponding closed symmetrized balls $\bar{B}_s(x,r)$. Then, one has [**12**, Sect. 4]:

Theorem 7.2. *For a standard stationary spacetime M, the following properties are equivalent:*

(1) *M is globally hyperbolic.*
(2) *The closed symmetrized balls $\bar{B}_s(x,r)$ of the Fermat metric F in (7.3) associated to one (and then to any) standard stationary splitting of M are compact.*

Moreover, the slices associated to a standard stationary splitting are Cauchy hypersurfaces if and only if the Fermat metric associated to that splitting is both, forward and backward complete.

Remark 7.3. For a standard static spacetime ($\delta \equiv 0$) the Fermat metric is just the reversible Finsler metric $F = \sqrt{g_0/\beta}$, and its associated distance is equal to the distance associated to the Riemannian metric g_0/β. So, one recovers the equivalence between:
(a) global hyperbolicity,
(b) completeness of g_0/β, and
(c) each S_t is Cauchy.
However, in the general stationary case the condition (b) splits into:
(b1) $\bar{B}_s(x,r)$ are compact, and
(b2) d_F is forward and backward complete.
One has only the equivalences (a)$\Leftrightarrow$ (b1), (b2)$\Leftrightarrow$ (c), plus the trivial implications (a) $\Leftarrow$ (c), (b1) $\Leftarrow$ (b2).

Finally, notice that any spacelike Cauchy hypersurface S' of a standard stationary space induces a standard stationary splitting such that S' is one of the levels $t =$ constant. So, all spacelike Cauchy hypersurfaces are characterized in terms of a Fermat metric.

Further properties of the Fermat metric in connection with both, the causal boundaries and other boundaries for Riemannian and Finsler manifolds, are being developed in [**19**].

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